

Regression analysis in quantum language

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Abstract

Although regression analysis has a great history, we consider that it has always continued being confused. For example, the fundamental terms in regression analysis (e.g., "regression", "least-squares method", "explanatory variable", "response variable", etc.) seem to be historically conventional, that is, these words do not express the essence of regression analysis. Recently, we proposed quantum language (or, classical and quantum measurement theory), which is characterized as the linguistic turn of the Copenhagen interpretation of quantum mechanics. We believe that this language has a great power of description, and therefore, even statistics can be described by quantum language. Therefore, in this paper, we discuss the regression analysis and the generalized linear model (i.e., multiple regression analysis) in quantum language, and clarify that the terms "explanatory variable" and "response variable" is respectively characterized as a kind of causality and the measured value.

Keywords: Copenhagen Interpretation, Operator Algebra, Quantum and Classical Measurement Theory, Fisher Maximum Likelihood Method, Regression Analysis, Generalized Linear Model

1 Introduction

1.1 The least-squared method in applied mathematics

Let us start from the simple explanation of the least-squared method. Let $\{(a_i, x_i)\}_{i=1}^n$ be a sequence in the two dimensional real space \mathbb{R}^2 . Let $\phi^{(\beta_1, \beta_2)} : \mathbb{R} \rightarrow \mathbb{R}$ be the simple function such that $\mathbb{R} \ni a \mapsto x = \phi^{(\beta_1, \beta_2)}(a) = \beta_1 a + \beta_0 \in \mathbb{R}$, where the pair $(\beta_1, \beta_2) \in \mathbb{R}^2$ is assumed to be unknown. Define the error σ by

$$\sigma^2(\beta_1, \beta_2) = \frac{1}{n} \sum_{i=1}^n (x_i - \phi^{(\beta_1, \beta_2)}(a_i))^2 \left(= \frac{1}{n} \sum_{i=1}^n (x_i - (\beta_1 a_i + \beta_0))^2 \right) \quad (1)$$

Then, we have the following minimization problem:

Problem 1 [The least-squared method].

(A) Find the $(\hat{\beta}_0, \hat{\beta}_1) \in \mathbb{R}^2$ such that

$$\sigma^2(\hat{\beta}_0, \hat{\beta}_1) = \min_{(\beta_1, \beta_2) \in \mathbb{R}^2} \sigma^2(\beta_1, \beta_2) \left(= \frac{1}{n} \min_{(\beta_1, \beta_2) \in \mathbb{R}^2} \sum_{i=1}^n (x_i - (\beta_1 a_i + \beta_0))^2 \right) \quad (2)$$

where $(\hat{\beta}_0, \hat{\beta}_1)$ is called "sample regression coefficients".

This is easily solved as follows. Taking partial derivatives with respect to β_0, β_1 , and equating the results to zero, gives the equations (i.e., "normal equations"),

$$\frac{\partial \sigma^2(\beta_1, \beta_2)}{\partial \beta_0} = \sum_{i=1}^n (x_i - \beta_0 - \beta_1 a_i) = 0, \quad (i = 1, \dots, n) \quad (3)$$

$$\frac{\partial \sigma^2(\beta_1, \beta_2)}{\partial \beta_1} = \sum_{i=1}^n (x_i - \beta_0 - \beta_1 a_i) a_i = 0, \quad (i = 1, \dots, n) \quad (4)$$

Solving it, we get that

$$\hat{\beta}_1 = \frac{s_{ax}}{s_{aa}}, \quad \hat{\beta}_0 = \bar{x} - \frac{s_{ax}}{s_{aa}}\bar{a}, \quad \hat{\sigma}^2 = \left(\frac{1}{n} \sum_{i=1}^n (x_i - (\hat{\beta}_1 a_i + \hat{\beta}_0))^2 \right) = s_{xx} - \frac{s_{ax}^2}{s_{aa}} \quad (5)$$

where

$$\bar{a} = \frac{a_1 + \cdots + a_n}{n}, \quad \bar{x} = \frac{x_1 + \cdots + x_n}{n}, \quad (6)$$

$$s_{aa} = \frac{(a_1 - \bar{a})^2 + \cdots + (a_n - \bar{a})^2}{n}, \quad s_{xx} = \frac{(x_1 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n}, \quad (7)$$

$$s_{ax} = \frac{(a_1 - \bar{a})(x_1 - \bar{x}) + \cdots + (a_n - \bar{a})(x_n - \bar{x})}{n}. \quad (8)$$

Remark 1 [Applied mathematics]. The above result is in applied mathematics and neither in statistics nor in quantum language. The purpose of this paper is to add a quantum linguistic story to Problem 1 (i.e., the least-squared method) in the framework of quantum language.

1.2 Quantum language (Axioms and Interpretation)

As mentioned in Remark 1, our purpose is to add a quantum linguistic story. Thus, we shall, according to ref. [10], mention the overview of quantum language (or, measurement theory, in short, MT).

Quantum language is characterized as the linguistic turn of the Copenhagen interpretation of quantum mechanics (cf. refs. [6], [13]). Quantum language (or, measurement theory) has two simple rules (i.e. Axiom 1 (concerning measurement) and Axiom 2 (concerning causal relation)) and the linguistic interpretation (= how to use the Axioms 1 and 2). That is,

$$(B_1) \quad \boxed{\text{Quantum language}} = \boxed{\text{Axiom 1}} + \boxed{\text{Axiom 2}} + \boxed{\text{linguistic interpretation}} \\ \text{(=MT(measurement theory))} \quad \text{(measurement)} \quad \text{(causality)} \quad \text{(how to use Axioms)}$$

(cf. refs. [3]- [11]). This is all of quantum language.

This theory is formulated in a certain C^* -algebra \mathcal{A} (cf. ref. [14]), and is classified as follows:

$$(B_2) \quad \text{Quantum language(=MT)} \begin{cases} \text{quantum MT} & \text{(when } \mathcal{A} \text{ is non-commutative)} \\ \text{classical MT} & \text{(when } \mathcal{A} \text{ is commutative, i.e., } \mathcal{A} = C_0(\Omega)) \end{cases}$$

where $C_0(\Omega)$ is the C^* -algebra composed of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space Ω .

Since our concern in this paper is concentrated to classical systems, we devote ourselves to the commutative C^* -algebra $C_0(\Omega)$, which is quite elementary. Therefore, we believe that all statisticians can understand our assertion (i.e., the quantum linguistic approach to statistics).

Let Ω is a locally compact Hausdorff space, which is also called a state space. And thus, an element $\omega (\in \Omega)$ is said to be a state. Let $C(\Omega)$ be the C^* -algebra composed of all bounded continuous complex-valued functions on a locally compact Hausdorff space Ω . The norm $\|\cdot\|_{C(\Omega)}$ is usual, i.e., $\|f\|_{C(\Omega)} = \sup_{\omega \in \Omega} |f(\omega)|$ ($\forall f \in C(\Omega)$).

Motivated by Davies' idea (cf. ref. [2]) in quantum mechanics, an observable $O = (X, \mathcal{F}, F)$ in $C_0(\Omega)$ (or, precisely, in $C(\Omega)$) is defined as follows:

(C₁) X is a topological space. $\mathcal{F} (\subseteq 2^X$ (i.e., the power set of X)) is a field, that is, it satisfies the following conditions (i)–(iii): (i): $\emptyset \in \mathcal{F}$, (ii): $\Xi \in \mathcal{F} \implies X \setminus \Xi \in \mathcal{F}$, (iii): $\Xi_1, \Xi_2, \dots, \Xi_n \in \mathcal{F} \implies \cup_{k=1}^n \Xi_k \in \mathcal{F}$.

(C₂) The map $F : \mathcal{F} \rightarrow C(\Omega)$ satisfies that

$$0 \leq [F(\Xi)](\omega) \leq 1, \quad [F(X)](\omega) = 1 \quad (\forall \omega \in \Omega)$$

and moreover, if

$$\Xi_1, \Xi_2, \dots, \Xi_k, \dots \in \mathcal{F}, \quad \Xi_m \cap \Xi_n = \emptyset \quad (m \neq n), \quad \Xi = \cup_{k=1}^{\infty} \Xi_k \in \mathcal{F},$$

then, it holds

$$[F(\Xi)](\omega) = \lim_{n \rightarrow \infty} \sum_{k=1}^n [F(\Xi_k)](\omega) \quad (\forall \omega \in \Omega)$$

Note that Hopf extension theorem (*cf.* ref. [15]) guarantees that $(X, \mathcal{F}, [F(\cdot)](\omega))$ is regarded as the mathematical probability space.

Example 1 [The normal observable]. Put $\Omega = \mathbb{R} \times \mathbb{R}_+ = \{(\mu, \sigma) \in \mathbb{R}^2 : \sigma > 0\}$. Define the normal observable $O_G = (\mathbb{R}, \mathcal{B}_{\mathbb{R}}, G)$ in $C_0(\mathbb{R} \times \mathbb{R}_+)$ such that

$$[G(\Xi)](\omega) = \frac{1}{\sqrt{2\pi}\sigma} \int_{\Xi} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx \quad (9)$$

$$(\forall \Xi \in \mathcal{B}_{\mathbb{R}} (= \text{Borel field in } \mathbb{R}), \quad \forall \omega = (\mu, \sigma) \in \Omega = \mathbb{R} \times \mathbb{R}_+).$$

This observable is the most fundamental in this paper.

Now we shall briefly explain "quantum language (B)" in classical systems as follows:

A measurement of an observable $O = (X, \mathcal{F}, F)$ for a system with a state $\omega (\in \Omega)$ is denoted by $M_{C_0(\Omega)}(O, S_{[\omega]})$. By the measurement, a measured value $x (\in X)$ is obtained as follows:

Axiom 1 [Measurement].

(D₁) *The probability that a measured value $x (\in X)$ obtained by the measurement $M_{C_0(\Omega)}(O \equiv (X, \mathcal{F}, F))$, $S_{[\omega_0]}$ belongs to a set $\Xi (\in \mathcal{F})$ is given by $[F(\Xi)](\omega_0)$.*

Axiom 2 [Causality].

(D₂) *The causality is represented by a Markov operator $\Phi_{21} : C_0(\Omega_2) \rightarrow C_0(\Omega_1)$. Particularly, the deterministic causality is represented by a continuous map $\phi_{12} : \Omega_1 \rightarrow \Omega_2$ such that*

$$f_2(\phi_{12}(\omega_1)) = [\Phi_{12}(f_2)](\omega_1) \quad (\forall f_2 \in C_0(\Omega_2), \omega_1 \in \Omega_1)$$

Also, see (21) later.

Interpretation [Linguistic interpretation]. Although there are several linguistic rules in quantum language, the following is the most important:

(D₃) *Only one measurement is permitted. And thus, the state is only one and does not move.*

In order to read this paper, it suffices to understand the above three. For the further arguments, see refs. [5], [6], [7], [8], [9], [10], [11].

Remark 2 [Random variables in Kolmogorov's probability theory]. It should be noted that the word of "random variable" in Kolmogorov's probability theory is not included in quantum language (i.e., Axioms 1 and 2). However, the theory of random variables (i.e., Kolmogorov's probability theory) is frequently used in the mathematical proofs of quantum linguistic statements, just like the mathematical theory of differential equations is used in the proofs of Newtonian mechanical statements. (Continued to Remark 3).

1.3 Fisher's maximum likelihood method (concerning Axiom 1)

It is usual to consider that we do not know the pure state $\omega_0 (\in \Omega)$ when we take a measurement $M_{C_0(\Omega)}(O, S_{[\omega_0]})$. That is because we usually take a measurement $M_{C_0(\Omega)}(O, S_{[\omega_0]})$ in order to know the state ω_0 . Thus, when we want to emphasize that we do not know the state ω_0 , $M_{C_0(\Omega)}(O, S_{[\omega_0]})$ is denoted by $M_{C_0(\Omega)}(O, S_{[*]})$.

Theorem 1 [Fisher's maximum likelihood method (cf. refs. [4], [5])]. Consider a measurement $M_{C_0(\Omega)}(O = (X, \mathcal{F}, F), S_{[*]})$. Assume that we know that the measured value $x (\in X)$ obtained by a measurement $M_{C_0(\Omega)}(O = (X, \mathcal{F}, F), S_{[*]})$ belongs to $\Xi(\in \mathcal{F})$. Then, there is a reason to infer that the unknown state $[*]$ is equal to $\omega_0(\in \Omega)$ such that

$$\min_{\omega_1 \in \Omega} \frac{[F(\Xi)](\omega_0)}{[F(\Xi)](\omega_1)} \left(= \frac{[F(\Xi)](\omega_0)}{\max_{\omega_1 \in \Omega} [F(\Xi)](\omega_1)} \right) = 1 \quad (10)$$

if the righthand side of this formula exists. Also, if $\Xi = \{x\}$, it suffices to calculate the $\omega_0(\in \Omega)$ such that

$$L(x, \omega_0) = 1 \quad (11)$$

where the likelihood function $L(x, \omega)(\equiv L_x(\omega))$ is defined by

$$L(x, \omega) = \inf_{\omega_1 \in \Omega} \left[\lim_{\Xi \supseteq \{x\}, [F(\Xi)](\omega_1) \neq 0, \Xi \rightarrow \{x\}} \frac{[F(\Xi)](\omega)}{[F(\Xi)](\omega_1)} \right] \quad (12)$$

Definition 1 [Product observable (or, simultaneous observable), simultaneous measurement]. For each $k = 1, 2, \dots, K$, consider an observable $O_k = (X_k, \mathcal{F}_k, F_k)$ in $C_0(\Omega)$. Define the simultaneous observable $\times_{k=1}^K O_k = (\times_{k=1}^K X_k, \boxtimes_{k=1}^K \mathcal{F}_k, \times_{k=1}^K F_k)$ in $C_0(\Omega)$ such that

$$\begin{aligned} \left(\times_{k=1}^K F_k \right) \left(\times_{k=1}^K \Xi_k \right) &= F_1(\Xi_1) F_2(\Xi_2) \cdots F_K(\Xi_K) \\ (\forall \Xi_k \in \mathcal{F}_k, \forall k = 1, \dots, K). \end{aligned} \quad (13)$$

where $\boxtimes_{k=1}^K \mathcal{F}_k$ is the product field of \mathcal{F}_k ($k = 1, 2, \dots, K$).

For each $k = 1, 2, \dots, K$, consider a measurement $M_{C_0(\Omega)}(O_k := (X_k, \mathcal{F}_k, F_k), S_{[\omega]})$. However, since the linguistic interpretation (D₃) says that only one measurement is permitted, the multiple measurements $\{M_{C_0(\Omega)}(O_k, S_{[\omega]})\}_{k=1}^K$ are prohibited. Thus, this $\{M_{C_0(\Omega)}(O_k, S_{[\omega]})\}_{k=1}^K$ is represented by the simultaneous measurement $M_{C_0(\Omega)}(\times_{k=1}^K O_k, S_{[\omega]})$.

Example 2 [Simultaneous normal observable]. Let $O_G = (\mathbb{R}, \mathcal{B}_{\mathbb{R}}, G)$ be the normal observable in $C_0(\mathbb{R} \times \mathbb{R}_+)$ in Example 1. Let n be a natural number. Then, we obtain the simultaneous normal observable $O_G^n = (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}}^n, G^n)$ in $C_0(\mathbb{R} \times \mathbb{R}_+)$. That is,

$$[G^n(\times_{i=1}^n \Xi_i)](\omega) = [G^n(\times_{i=1}^n \Xi_i)](\mu, \sigma) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \int \cdots \int \exp\left[-\frac{\sum_{i=1}^n (x_i - \mu)^2}{2\sigma^2}\right] dx_1 dx_2 \cdots dx_n \quad (14)$$

$$\times_{i=1}^n \Xi_i$$

$$(\forall \Xi_i \in \mathcal{B}_{\mathbb{R}} (= \text{Borel field in } \mathbb{R}), (i = 1, 2, \dots, n), \quad \forall \omega = (\mu, \sigma) \in \Omega = \mathbb{R} \times \mathbb{R}_+).$$

Thus, we have the simultaneous measurement $M_{C_0(\mathbb{R} \times \mathbb{R}_+)}(O_G^n = (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}}^n, G^n), S_{[*]})$ in $C_0(\mathbb{R} \times \mathbb{R}_+)$. Assume that a measured value $x = (x_1, x_2, \dots, x_n) (\in \mathbb{R}^n)$ is obtained by the measurement. Since the likelihood function $L_x(\mu, \sigma) (= L(x, (\mu, \sigma)))$ is defined by

$$L_x(\mu, \sigma) = \frac{C}{(\sqrt{2\pi}\sigma)^n} \exp\left[-\frac{\sum_{i=1}^n (x_i - \mu)^2}{2\sigma^2}\right] \quad (\text{the constant } C \text{ is independent of } \mu, \sigma) \quad (15)$$

it suffices to calculate the following equations:

$$\frac{\partial L_x(\mu, \sigma)}{\partial \mu} = 0, \quad \frac{\partial L_x(\mu, \sigma)}{\partial \sigma} = 0 \quad (16)$$

Thus, we see, by Theorem 1 (Fisher's maximum likelihood method), that the unknown state $[*]$ can be inferred by $(\hat{\mu}, \hat{\sigma})$, that is,

$$\hat{\mu}(x) = \hat{\mu}(x_1, x_2, \dots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n} \quad (17)$$

$$\hat{\sigma}(x) = \hat{\sigma}(x_1, x_2, \dots, x_n) = \sqrt{\frac{\sum_{i=1}^n (x_i - \hat{\mu}(x))^2}{n}} \quad (18)$$

For example, consider the following image observable: $\hat{\mu}(\mathcal{O}_G^n) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}}, G^n \circ \hat{\mu}^{-1})$ in $C_0(\mathbb{R} \times \mathbb{R}_+)$ such that

$$[(G^n \circ \hat{\mu}^{-1})(\Xi_1)](\omega) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \int_{\{x \in \mathbb{R}^n : \hat{\mu}(x) \in \Xi_1\}} \cdots \int \exp\left[-\frac{\sum_{i=1}^n (x_i - \mu)^2}{2\sigma^2}\right] dx_1 dx_2 \cdots dx_n \quad (19)$$

which is calculated as follows:

$$= \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_{\Xi_1} \exp\left[-\frac{n(x - \mu)^2}{2\sigma^2}\right] dx \quad (20)$$

($\forall \omega = (\mu, \sigma) \in \Omega \equiv \mathbb{R} \times \mathbb{R}_+, \forall \Xi_1 \in \mathcal{B}_{\mathbb{R}}$).

Remark 3 [Kolmogorov's probability theory]. Although the derivation of (20) from (19) may not be easy, it is the problem in mathematics. Although there are several derivations, the calculation in the framework of Kolmogorov's probability theory (ref. [12]) may be the most elegant. Thus, mathematical theories (e.g., Kolmogorov's probability theory, operator theory (ref. [14])) are frequently used in quantum language.

1.4 The Heisenberg picture (concerning Axiom 2)

Consider a tree-like ordered set $(T := \{t_0, t_1, \dots, t_n\}, \leq)$ with the root t_0 (i.e., $t_0 \leq t$ ($\forall t \in T$)). This is also characterized by the parent map $\tau : T \setminus \{t_0\} \rightarrow T$ such that $\tau(t) = \max\{s \in T \mid s < t\}$. Put $T_{\leq}^2 = \{(t, t') \in T^2 : t \leq t'\}$. In Figure 1, see the root t_0 , the parent map: $\tau(t_3) = \tau(t_4) = t_2$, $\tau(t_2) = \tau(t_5) = t_1$, $\tau(t_1) = \tau(t_6) = \tau(t_7) = t_0$

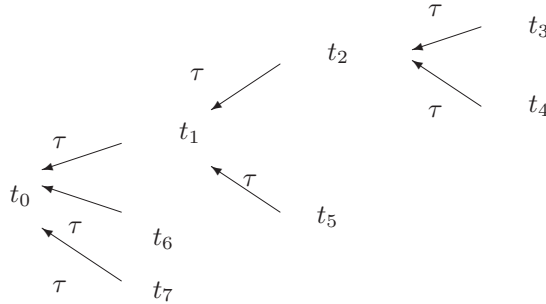


Figure 1: Tree

For each $t \in T$, a commutative C^* -algebra $C_0(\Omega_t)$ is associated. According to Axiom 2, consider a Markov relation (i.e., causal relation) $\{\Phi_{t,t'} : C_0(\Omega_{t'}) \rightarrow C_0(\Omega_t)\}_{(t,t') \in T_{\leq}^2}$, which is also represented by $\{\Phi_{\tau(t),t} : C_0(\Omega_t) \rightarrow C_0(\Omega_{\tau(t)})\}_{t \in T \setminus \{t_0\}}$. In this paper, we consider the deterministic case, that is, the case that $\Phi_{\tau(t),t} : C_0(\Omega_t) \rightarrow C_0(\Omega_{\tau(t)})$ is represented by the continuous map (called "causal map") $\phi_{\tau(t),t} : \Omega_{\tau(t)} \rightarrow \Omega_t$ such as

$$(\Phi_{\tau(t),t} f)(\omega_{\tau(t)}) = f(\phi_{\tau(t),t}(\omega_{\tau(t)})) \quad (\forall \omega_{\tau(t)} \in \Omega_{\tau(t)}, \forall f \in C_0(\Omega_t), \forall t \in T \setminus \{t_0\}) \quad (21)$$

Let an observable $\mathcal{O}_t := (X_t, \mathcal{F}_t, F_t)$ in the $C_0(\Omega_t)$ be given for each $t \in T$. $\Phi_{\tau(t),t} \mathcal{O}_t$ is defined by $(X_t, \mathcal{F}_t, \Phi_{\tau(t),t} F_t)$ in the $C_0(\Omega_{\tau(t)})$. And let $\omega_0 \in \Omega_{t_0}$. Consider "measurements" such as

- (E) for each $t \in T$, take a measurement of an observable \mathcal{O}_t for the system with a "moving state" $\phi_{t_0,t}(\omega_0) \in \Omega_t$.

where the meaning of "moving state" is not clear yet. Recalling that the linguistic interpretation (D₃) says that a state never moves, we consider the meaning of the (E) as follows: For each $s \in T$, put $T_s = \{t \in T \mid t \geq s\}$. And define the observable $\hat{\mathcal{O}}_s = (\times_{t \in T_s} X_t, \boxtimes_{t \in T_s} \mathcal{F}_t, \hat{F}_s)$ in $C_0(\Omega_s)$ (due to the Heisenberg picture) as follows:

$$\widehat{\mathbf{O}}_s = \begin{cases} \mathbf{O}_s & (\text{if } s \in T \setminus \tau(T)) \\ \mathbf{O}_s \times (\times_{t \in \tau^{-1}(\{s\})} \Phi_{\tau(t), t} \widehat{\mathbf{O}}_t) & (\text{if } s \in \tau(T)) \end{cases} \quad (22)$$

Using (22) iteratively, we can finally obtain the observable $\widehat{\mathbf{O}}_{t_0}$ in $C_0(\Omega_{t_0})$. Thus, the above (E) is represented by the measurement $\mathbf{M}_{C_0(\Omega_{t_0})}(\widehat{\mathbf{O}}_{t_0}, S_{[\omega_0]})$. Since the causal map is assumed to be deterministic in this paper, the $\widehat{\mathbf{O}}_{t_0}$ is simply represented by the simultaneous observable such as $\widehat{\mathbf{O}}_{t_0} = \times_{t \in T} \Psi_{t_0, t} \mathbf{O}_t$ (cf. refs. [5], [8], [9]).

Remark 4 [What is regression analysis?]. Since regression analysis has various aspects, it is not easy to answer the question: "What is regression analysis?" However, we can say that regression analysis is at least related to the inference concerning $\mathbf{M}_{C_0(\Omega_{t_0})}(\widehat{\mathbf{O}}_{t_0}, S_{[*]})$. In this sense, regression analysis must be related to Axiom 2 as well as Axiom 1. On the other hand, Fisher's maximum likelihood method is related to only Axiom 1. We believe that the reason that regression analysis is famous is to be related to Axiom 2. As seen in (B₁), the importance of Axiom 2 (Causality) is explicitly emphasized in quantum language and not in statistics. Thus, we think that regression analysis plays the role of Axiom 2 in the conventional statistics. And, in Section 2, we will point out that the term "explanatory variable" is understood as a kind of causal map in quantum language.

1.5 The reverse relation between confidence interval and statistical hypothesis testing

Let $\mathbf{O} = (X, \mathcal{F}, F)$ be an observable formulated in a commutative C^* -algebra $C_0(\Omega)$. Let X be a topological space. Let Θ be a locally compact space with the semi-distance d_Θ^x ($\forall x \in X$), that is, for each $x \in X$, the map $d_\Theta^x : \Theta^2 \rightarrow [0, \infty)$ satisfies that (i): $d_\Theta^x(\theta, \theta) = 0$, (ii): $d_\Theta^x(\theta_1, \theta_2) = d_\Theta^x(\theta_2, \theta_1)$, (iii): $d_\Theta^x(\theta_1, \theta_3) \leq d_\Theta^x(\theta_1, \theta_2) + d_\Theta^x(\theta_2, \theta_3)$.

Let $\widehat{E} : X \rightarrow \Theta$ and $\pi : \Omega \rightarrow \Theta$ be continuous maps, which are respectively called an estimator and a quantity. Let α be a real number such that $0 < \alpha \ll 1$, for example, $\alpha = 0.05$. For any state $\omega (\in \Omega)$, define the positive number $\eta_\omega^\alpha (> 0)$ such that:

$$\begin{aligned} \eta_\omega^\alpha &= \inf\{\eta > 0 : [F(\{x \in X : d_\Theta^x(\widehat{E}(x), \pi(\omega)) \geq \eta\})](\omega) \leq \alpha\} \\ &\left(= \inf\{\eta > 0 : [F(\{x \in X : d_\Theta^x(\widehat{E}(x), \pi(\omega)) < \eta\})](\omega) \geq 1 - \alpha\} \right) \end{aligned} \quad (23)$$

Then Axiom 1 says that:

(F₁) *the probability, that the measured value x obtained by the measurement $\mathbf{M}_{C_0(\Omega)}(\mathbf{O} := (X, \mathcal{F}, F), S_{[\omega_0]})$ satisfies the following condition (24), is more than or equal to $1 - \alpha$ (e.g., $1 - \alpha = 0.95$).*

$$d_\Theta^x(\widehat{E}(x), \pi(\omega_0)) < \eta_{\omega_0}^\alpha \quad (24)$$

or equivalently,

(F₂) *the probability, that the measured value x obtained by the measurement $\mathbf{M}_{C_0(\Omega)}(\mathbf{O} := (X, \mathcal{F}, F), S_{[\omega_0]})$ satisfies the following condition (25), is less than or equal to α (e.g., $\alpha = 0.05$).*

$$d_\Theta^x(\widehat{E}(x), \pi(\omega_0)) \geq \eta_{\omega_0}^\alpha \quad (25)$$

Theorem 2 [Confidence interval and statistical hypothesis testing (cf. ref. [10])]. *Let $\mathbf{O} = (X, \mathcal{F}, F)$ be an observable formulated in a commutative C^* -algebra $C_0(\Omega)$. Let $\widehat{E} : X \rightarrow \Theta$ and $\pi : \Omega \rightarrow \Theta$ be an estimator and a quantity respectively. Let η_ω^α be as defined in the formula (23).*

From the (F₁), we assert "the confidence interval method" as follows:

(G₁) [The confidence interval method]. *For any $x \in X$, define*

$$I_x^{1-\alpha} = \{\pi(\omega) (\in \Theta) : d_\Theta^x(\widehat{E}(x), \pi(\omega)) < \eta_\omega^{1-\alpha}\} \quad (26)$$

which is called the $(1 - \alpha)$ -confidence interval. Let $x(\in X)$ be a measured value x obtained by the measurement $M_{C_0(\Omega)}(\mathbf{O} := (X, \mathcal{F}, F), S_{[\omega_0]})$. Then, the probability that $I_x^{1-\alpha} \ni \pi(\omega_0)$ is more than or equal to $1 - \alpha$.

From the (F_2) , we assert "the statistical hypothesis test" as follows:

(G₂) [The statistical hypothesis test]. Assume that a state ω_0 satisfies that $\pi(\omega_0) \in H_N(\subseteq \Theta)$, where H_N is called a "null hypothesis". Put

$$\hat{R}_{H_N}^{\alpha; \Theta} = \bigcap_{\omega \in \Omega \text{ such that } \pi(\omega) \in H_N} \{\hat{E}(x)(\in \Theta) : d_{\Theta}^x(\hat{E}(x), \pi(\omega)) \geq \eta_{\omega}^{\alpha}\} \quad (27)$$

and also

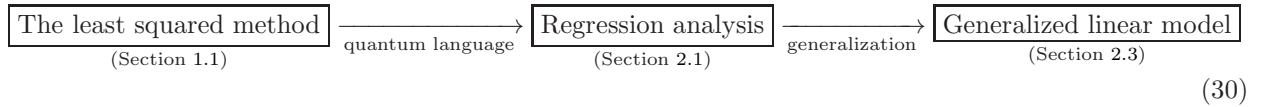
$$\hat{R}_{H_N}^{\alpha; X} = \hat{E}^{-1}(\hat{R}_{H_N}^{\alpha; \Theta}) = \bigcap_{\omega \in \Omega \text{ such that } \pi(\omega) \in H_N} \{x(\in X) : d_{\Theta}^x(\hat{E}(x), \pi(\omega)) \geq \eta_{\omega}^{\alpha}\} \quad (28)$$

which is respectively called the (α) -rejection region of the null hypothesis H_N . Then, the probability, that the measured value $x(\in X)$ obtained by the measurement $M_{C_0(\Omega)}(\mathbf{O} := (X, \mathcal{F}, F), S_{[\omega_0]})$ (where it should be noted that $\pi(\omega_0) \in H_N$) satisfies the following condition (29), is less than or equal to α (e.g., $\alpha = 0.05$).

$$"\hat{E}(x) \in \hat{R}_{H_N}^{\alpha; \Theta}" \text{ or equivalently } "x \in \hat{R}_{H_N}^{\alpha; X}" \quad (29)$$

2 Regression analysis in quantum language

In this section, we show that the least squared method (mentioned in Section 1.1) acquires a quantum linguistic story as follows.



Note that Theorem 1 (Fisher's maximum likelihood method) and Theorem 2 (Confidence interval and hypothesis test) are only related to Axiom 1. On the other hand, it should be noted that Axiom 2 (as well as Axiom 1) is used in regression analysis.

2.1 Simple regression analysis in quantum language

Put $T = \{0, 1, 2, \dots, i, \dots, n\}$. And let $(T, \tau : T \setminus \{0\} \rightarrow T)$ be the tree-like ordered set (with the parallel structure) such that

$$\tau(i) = 0 \quad (\forall i = 1, 2, \dots, n) \quad (31)$$

For each $i \in T$, define a locally compact space Ω_i such that

$$\Omega_0 = \mathbb{R}^2 = \left\{ \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} : \beta_0, \beta_1 \in \mathbb{R} \right\} \quad (32)$$

$$\Omega_i = \mathbb{R} = \left\{ \mu_i : \mu_i \in \mathbb{R} \right\} \quad (i = 1, 2, \dots, n) \quad (33)$$

Assume that

$$a_i \in \mathbb{R} \quad (i = 1, 2, \dots, n), \quad (34)$$

which are called *explanatory variables* in the conventional statistics. Consider the deterministic causal map $\psi_{a_i} : \Omega_0(= \mathbb{R}^2) \rightarrow \Omega_i(= \mathbb{R})$ such that

$$\Omega_0 = \mathbb{R}^2 \ni \beta = (\beta_0, \beta_1) \mapsto \psi_{a_i}(\beta_0, \beta_1) = \beta_0 + \beta_1 a_i = \mu_i \in \Omega_i = \mathbb{R} \quad (35)$$

which is equivalent to the deterministic Markov operator $\Psi_{a_i} : C_0(\Omega_i) \rightarrow C_0(\Omega_0)$ such that

$$[\Psi_{a_i}(f_i)](\omega_0) = f_i(\psi_{a_i}(\omega_0)) \quad (\forall f_i \in C_0(\Omega_i), \forall \omega_0 \in \Omega_0, \forall i \in 1, 2, \dots, n) \quad (36)$$

Thus, under the identification: $a_i \Leftrightarrow \Psi_{a_i}$, the term "explanatory variable" means a kind of causal relation Ψ_{a_i} .

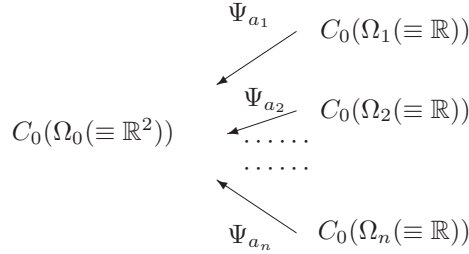


Figure 2: Parallel structure (Causal relation Ψ_{a_i})

For each $i = 1, 2, \dots, n$, define the *normal observable* $O_i \equiv (\mathbb{R}, \mathcal{B}_{\mathbb{R}}, G_{\sigma})$ in $C_0(\Omega_i(= \mathbb{R}))$ such that

$$[G_{\sigma}(\Xi)](\mu) = \frac{1}{(\sqrt{2\pi\sigma^2})} \int_{\Xi} \exp \left[-\frac{(x - \mu)^2}{2\sigma^2} \right] dx \quad (\forall \Xi \in \mathcal{B}_{\mathbb{R}}, \forall \mu \in \Omega_i(= \mathbb{R})) \quad (37)$$

where σ is a positive constant.

Thus, we have the observable $O_0^{a_i} \equiv (\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \Psi_{a_i} G_{\sigma})$ in $C_0(\Omega_0(= \mathbb{R}^2))$ such that

$$[\Psi_{a_i}(G_{\sigma}(\Xi))](\beta) = [(G_{\sigma}(\Xi))](\psi_{a_i}(\beta)) = \frac{1}{(\sqrt{2\pi\sigma^2})} \int_{\Xi} \exp \left[-\frac{(x - (\beta_0 + a_i \beta_1))^2}{2\sigma^2} \right] dx \quad (38)$$

$$(\forall \Xi \in \mathcal{B}_{\mathbb{R}}, \forall \beta = (\beta_0, \beta_1) \in \Omega_0(= \mathbb{R}^2))$$

Hence, we have the simultaneous observable $\times_{i=1}^n O_0^{a_i} \equiv (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, \times_{i=1}^n \Psi_{a_i} G_{\sigma})$ in $C_0(\Omega_0(= \mathbb{R}^2))$ such that

$$\begin{aligned} & [(\times_{i=1}^n \Psi_{a_i} G_{\sigma})(\times_{i=1}^n \Xi_i)](\beta) = \times_{i=1}^n ([\Psi_{a_i} G_{\sigma}(\Xi_i)](\beta)) \\ & = \frac{1}{(\sqrt{2\pi\sigma^2})^n} \int \cdots \int \exp \left[-\frac{\sum_{i=1}^n (x_i - (\beta_0 + a_i \beta_1))^2}{2\sigma^2} \right] dx_1 \cdots dx_n \\ & \quad \times_{i=1}^n \Xi_i \\ & = \int \cdots \int p_{(\beta_0, \beta_1, \sigma)}(x_1, x_2, \dots, x_n) dx_1 \cdots dx_n \\ & \quad \times_{i=1}^n \Xi_i \\ & \quad (\forall \times_{i=1}^n \Xi_i \in \mathcal{B}_{\mathbb{R}^n}, \forall \beta = (\beta_0, \beta_1) \in \Omega_0(= \mathbb{R}^2)) \end{aligned} \quad (39)$$

Assuming that σ is variable, we have the observable $O = (\mathbb{R}^n(= X), \mathcal{B}_{\mathbb{R}^n}(= \mathcal{F}), F)$ in $C_0(\Omega_0 \times \mathbb{R}_+)$ such that

$$[F(\times_{i=1}^n \Xi_i)](\beta, \sigma) = [(\times_{i=1}^n \Psi_{a_i} G_{\sigma})(\times_{i=1}^n \Xi_i)](\beta) \quad (\forall \Xi_i \in \mathcal{B}_{\mathbb{R}}, \forall (\beta, \sigma) \in \mathbb{R}^2(= \Omega_0) \times \mathbb{R}_+) \quad (40)$$

Problem 2 [Simple regression analysis in quantum language] Assume that a measured value $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in$

$X = \mathbb{R}^n$ is obtained by the measurement $\mathbf{M}_{C_0(\Omega_0 \times \mathbb{R}_+)}(\mathbf{O} \equiv (X, \mathcal{F}, F), S_{[(\beta_0, \beta_1, \sigma)]})$. We do not know the state $(\beta_0, \beta_1, \sigma^2)$. Then, from the measured value $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, infer the β_0, β_1, σ ! That is, represent the $(\beta_0, \beta_1, \sigma)$ by $(\hat{\beta}_0(x), \hat{\beta}_1(x), \hat{\sigma}(x))$ (i.e., the functions of x).

Answer. Taking partial derivatives with respect to $\beta_0, \beta_1, \sigma^2$, and equating the results to zero, gives the log-likelihood equations. That is, putting $L(\beta_0, \beta_1, \sigma^2, x_1, x_2, \dots, x_n) = \log p_{(\beta_0, \beta_1, \sigma)}(x_1, x_2, \dots, x_n)$, we see that

$$\frac{\partial L}{\partial \beta_0} = 0 \implies \sum_{i=1}^n (x_i - (\beta_0 + a_i \beta_1)) = 0 \quad (41)$$

$$\frac{\partial L}{\partial \beta_1} = 0 \implies \sum_{i=1}^n a_i (x_i - (\beta_0 + a_i \beta_1)) = 0 \quad (42)$$

$$\frac{\partial L}{\partial \sigma^2} = 0 \implies -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^n (x_i - \beta_0 - \beta_1 a_i)^2 = 0 \quad (43)$$

Therefore, using the notations (6)-(8), we obtain that

$$\hat{\beta}_0(x) = \bar{x} - \hat{\beta}_1(x) \bar{a} = \bar{x} - \frac{s_{ax}}{s_{aa}} \bar{a}, \quad \hat{\beta}_1(x) = \frac{s_{ax}}{s_{aa}} \quad (44)$$

and

$$\begin{aligned} (\hat{\sigma}(x))^2 &= \frac{\sum_{i=1}^n (x_i - (\hat{\beta}_0(x) + a_i \hat{\beta}_1(x)))^2}{n} \\ &= \frac{\sum_{i=1}^n \left(x_i - \left(\bar{x} - \frac{s_{ax}}{s_{aa}} \bar{a} \right) - a_i \frac{s_{ax}}{s_{aa}} \right)^2}{n} = \frac{\sum_{i=1}^n \left((x_i - \bar{x}) + (\bar{a} - a_i) \frac{s_{ax}}{s_{aa}} \right)^2}{n} \\ &= s_{xx} - 2s_{ax} \frac{s_{ax}}{s_{aa}} + s_{aa} \left(\frac{s_{ax}}{s_{aa}} \right)^2 = s_{xx} - \frac{s_{ax}^2}{s_{aa}} \end{aligned} \quad (45)$$

Note that the above (44) and (45) are the same as (5). Therefore, Problem 2 (i.e., regression analysis in quantum language) is a quantum linguistic stories of the least squared method (Problem 1).

2.2 Several properties (Distributions, confidence interval and hypothesis test)

Since our main assertion is to mention Problem 1, this section may be regarded as a kind of appendix. For the detailed proofs of Lemma 1, see standard books of statistics (e.g., ref. [1]).

Let $\mathbf{M}_{C_0(\Omega_0 \times \mathbb{R}_+)}(\mathbf{O} \equiv (X (= \mathbb{R}^n), \mathcal{F}, F), S_{[(\beta_0, \beta_1, \sigma)]})$ be the observable in Problem 3. For each $(\beta, \sigma) \in \mathbb{R}^2 \times \mathbb{R}_+$, we have the probability space $(X, \mathcal{F}, P_{(\beta, \sigma)})$, where $P_{(\beta, \sigma)}(\Xi) = F(\Xi)(\beta_0, \beta_1, \sigma)$ ($\forall \Xi \in \mathcal{F}$).

Put

$$L^2(X) = \{ \text{measurable function } f : X \rightarrow \mathbb{R} \mid [\int_X |f(x)|^2 P_{(\beta, \sigma)}(dx)]^{1/2} < \infty \}.$$

For any $f, g \in L^2(X)$, define $E(f)$ and $V(f)$ such that

$$E(f) = \int_X f(x) P_{(\beta, \sigma)}(dx), \quad V(f) = \int_X |f(x) - E(f)|^2 P_{(\beta, \sigma)}(dx). \quad (46)$$

Lemma 1 Consider the measurement $\mathbf{M}_{C_0(\Omega_0 \times \mathbb{R}_+)}(\mathbf{O} \equiv (X, \mathcal{F}, F), S_{[(\beta_0, \beta_1, \sigma)]})$ in Problem 3. And assume the above notations. Then, we see:

$$(H_1) \text{ (1): } V(\hat{\beta}_0) = \frac{\sigma^2}{n} \left(1 + \frac{\bar{a}^2}{s_{aa}}\right), \quad (2): V(\hat{\beta}_1) = \frac{\sigma^2}{n} \frac{1}{s_{aa}},$$

(H₂) [Studentization]. Motivated by the (H₁), we see:

$$T_{\beta_0} := \frac{\sqrt{n}(\hat{\beta}_0 - \beta_0)}{\sqrt{\hat{\sigma}^2(1 + \bar{a}^2/s_{aa})}} \sim t_{n-2}, \quad T_{\beta_1} := \frac{\sqrt{n}(\hat{\beta}_1 - \beta_1)}{\sqrt{\hat{\sigma}^2/s_{aa}}} \sim t_{n-2} \quad (47)$$

where t_{n-2} is the student's distribution with $n - 2$ degrees of freedom.

For the proof. see ref. [1].

Let $M_{C_0(\Omega_0(=\mathbb{R}^2) \times \mathbb{R}_+)}(\mathbf{O} \equiv (X(=\mathbb{R}^n), \mathcal{F}, F), S_{[(\beta_0, \beta_1, \sigma)]})$ be the observable in Problem 2. For each $k = 0, 1$, define the estimator $\hat{E}_k : X(=\mathbb{R}^n) \rightarrow \Theta_k(=\mathbb{R})$ and the quantity $\pi_k : \Omega(=\mathbb{R}^2 \times \mathbb{R}_+) \rightarrow \Theta_k(=\mathbb{R})$ as follows.

$$\begin{aligned} \hat{E}_0(x)(=\hat{\beta}_0(x)) &= \bar{x} - \frac{s_{ax}}{s_{aa}}\bar{a}, \quad \hat{E}_1(x)(=\hat{\beta}_1(x)) = \frac{s_{ax}}{s_{aa}}, \quad \pi_0(\beta_0, \beta_1, \sigma) = \beta_0, \quad \pi_1(\beta_0, \beta_1, \sigma) = \beta_1, \\ (\forall (\beta_0, \beta_1, \sigma) \in \mathbb{R}^2 \times \mathbb{R}_+) \end{aligned} \quad (48)$$

Let α be a real number such that $0 < \alpha \ll 1$, for example, $\alpha = 0.05$. For any state $\omega = (\beta, \sigma)(\in \Omega = \mathbb{R}^2 \times \mathbb{R}_+)$, define the positive number $\eta_{\omega, k}^\alpha (> 0)$ by (23), that is,

$$\eta_{\omega, k}^\alpha = \inf\{\eta > 0 : [F(\{x \in X : d_{\Theta_k}^x(\hat{E}_k(x), \pi_k(\omega)) \geq \eta\})](\omega) \leq \alpha\} \quad (49)$$

where, for each $\theta_k^0, \theta_k^1(\in \Theta_k)$, the semi-distance $d_{\Theta_k}^x$ in Θ_k is defined by

$$d_{\Theta_k}^x(\theta_k^0, \theta_k^1) = \begin{cases} \frac{\sqrt{n}|\theta_k^0 - \theta_k^1|}{\sqrt{\hat{\sigma}^2(1 + \bar{a}^2/s_{aa})}} & (\text{if } k = 0) \\ \frac{\sqrt{n}|\theta_k^0 - \theta_k^1|}{\sqrt{\hat{\sigma}^2/s_{aa}}} & (\text{if } k = 1) \end{cases} \quad (50)$$

Therefore, we see, by Lemma 1, that

$$\eta_{\omega, k}^\alpha = \begin{cases} \inf\{\eta > 0 : [F(\{x \in X : \frac{\sqrt{n}|\hat{\beta}_0(x) - \beta_0|}{\sqrt{\hat{\sigma}^2(1 + \bar{a}^2/s_{aa})}} \geq \eta\})](\omega) \leq \alpha\} & (\text{if } k = 0) \\ \inf\{\eta > 0 : [F(\{x \in X : \frac{\sqrt{n}|\hat{\beta}_1(x) - \beta_1|}{\sqrt{\hat{\sigma}^2(x)/s_{aa}}} \geq \eta\})](\omega) \leq \alpha\} & (\text{if } k = 1) \end{cases} \quad (51)$$

$$= t_{n-2}(\alpha/2) \quad (52)$$

The following propositions (described in quantum language) immediately follow from (52).

Proposition 1 [Confidence interval]. Assume that a measured value $x \in X$ is obtained by the measurement $M_{C_0(\Omega_0 \times \mathbb{R}_+)}(\mathbf{O} \equiv (X, \mathcal{F}, F), S_{[(\beta_0, \beta_1, \sigma)]})$. Here, the state $(\beta_0, \beta_1, \sigma)$ is assumed to be unknown. Then, we have the $(1 - \alpha)$ -confidence interval $I_{x, k}^{1-\alpha}$ in Theorem 2 as follows.

$$\begin{aligned} I_{x, k}^{1-\alpha} &= \{\pi_k(\omega)(\in \Theta_k) : d_{\Theta_k}^x(\hat{E}_k(x), \pi_k(\omega)) < \eta_{\omega, k}^{1-\alpha}\} \\ &= \begin{cases} I_{x, 0}^{1-\alpha} = \left\{ \beta_0 = \pi_0(\omega)(\in \Theta_0) : \frac{|\hat{\beta}_0(x) - \beta_0|}{\sqrt{\frac{\hat{\sigma}^2(x)}{n}(1 + \bar{a}^2/s_{aa})}} \leq t_{n-2}(\alpha/2) \right\} & (\text{if } k = 0) \\ I_{x, 1}^{1-\alpha} = \left\{ \beta_1 = \pi_1(\omega)(\in \Theta_1) : \frac{|\hat{\beta}_1(x) - \beta_1|}{\sqrt{\frac{\hat{\sigma}^2(x)}{n}(1/s_{aa})}} \leq t_{n-2}(\alpha/2) \right\} & (\text{if } k = 1) \end{cases} \end{aligned} \quad (53)$$

Proposition 2 [Hypothesis test]. Consider the measurement $M_{C_0(\Omega_0 \times \mathbb{R}_+)}(\mathbf{O} \equiv (X, \mathcal{F}, F), S_{[(\beta_0, \beta_1, \sigma)]})$. Here, the state $(\beta_0, \beta_1, \sigma)$ is assumed to be unknown. Then, according to Theorem 2, we say:

(I₁) Assume the null hypothesis $H_N = \{\beta_0\}(\subseteq \Theta_0 = \mathbb{R})$. Then, the rejection region is as follows:

$$\begin{aligned}\hat{R}_{H_N}^{\alpha;X} &= \hat{E}_0^{-1}(\hat{R}_{H_N}^{\alpha;\Theta_0}) = \bigcap_{\omega \in \Omega \text{ such that } \pi_0(\omega) \in H_N} \{x(\in X) : d_{\Theta_0}^x(\hat{E}_0(x), \pi_0(\omega)) \geq \eta_\omega^\alpha\} \\ &= \left\{x \in X : \frac{|\hat{\beta}_0(x) - \beta_0|}{\sqrt{\frac{\hat{\sigma}^2(x)}{n}(1 + \bar{a}^2/s_{aa})}} \geq t_{n-2}(\alpha/2)\right\}\end{aligned}\quad (54)$$

(I₂) Assume the null hypothesis $H_N = \{\beta_1\}(\subseteq \Theta_1 = \mathbb{R})$. Then, the rejection region is as follows:

$$\begin{aligned}\hat{R}_{H_N}^{\alpha;X} &= \hat{E}_1^{-1}(\hat{R}_{H_N}^{\alpha;\Theta_1}) = \bigcap_{\omega \in \Omega \text{ such that } \pi_1(\omega) \in H_N} \{x(\in X) : d_{\Theta_1}^x(\hat{E}_1(x), \pi_1(\omega)) \geq \eta_\omega^\alpha\} \\ &= \left\{x \in X : \frac{|\hat{\beta}_1(x) - \beta_1|}{\sqrt{\frac{\hat{\sigma}^2(x)}{n}(1/s_{aa})}} \geq t_{n-2}(\alpha/2)\right\}\end{aligned}\quad (55)$$

2.3 The quantum linguistic formulation of generalized linear model

As the generalization of Section 2.1, we shall discuss the generalized linear model in quantum language as follows:

Put $T = \{0, 1, 2, \dots, i, \dots, n\}$, which is the same as the tree (31), that is,

$$\tau(i) = 0 \quad (\forall i = 1, 2, \dots, n) \quad (56)$$

For each $i \in T$, define a locally compact space Ω_i such that

$$\Omega_0 = \mathbb{R}^{m+1} = \left\{\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} : \beta_0, \beta_1, \dots, \beta_m \in \mathbb{R}\right\} \quad (57)$$

$$\Omega_i = \mathbb{R} = \left\{\mu_i : \mu_i \in \mathbb{R}\right\} \quad (i = 1, 2, \dots, n) \quad (58)$$

Assume that

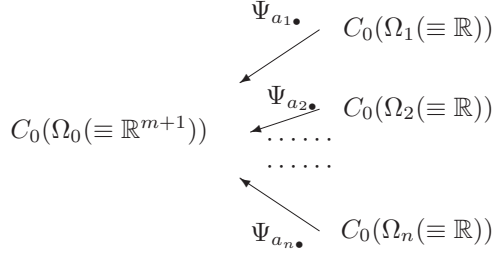
$$a_{ij} \in \mathbb{R} \quad (i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m, (m+1 \leq n)), \quad (59)$$

which are called *explanatory variables* in the conventional statistics. Consider the deterministic causal map $\psi_{a_{i\bullet}} : \Omega_0 (= \mathbb{R}^{m+1}) \rightarrow \Omega_i (= \mathbb{R})$ such that

$$\begin{aligned}\Omega_0 = \mathbb{R}^{m+1} \ni \beta = (\beta_0, \beta_1, \dots, \beta_m) &\mapsto \psi_{a_{i\bullet}}(\beta_0, \beta_1, \dots, \beta_m) = \beta_0 + \sum_{j=1}^m \beta_j a_{ij} = \mu_i \in \Omega_i = \mathbb{R} \\ &(i = 1, 2, \dots, n)\end{aligned}\quad (60)$$

Summing up, we see

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} \mapsto \begin{bmatrix} \psi_{a_{1\bullet}}(\beta_0, \beta_1, \dots, \beta_m) \\ \psi_{a_{2\bullet}}(\beta_0, \beta_1, \dots, \beta_m) \\ \psi_{a_{3\bullet}}(\beta_0, \beta_1, \dots, \beta_m) \\ \vdots \\ \psi_{a_{n\bullet}}(\beta_0, \beta_1, \dots, \beta_m) \end{bmatrix} = \begin{bmatrix} 1 & a_{11} & a_{12} & \cdots & a_{1m} \\ 1 & a_{21} & a_{22} & \cdots & a_{2m} \\ 1 & a_{31} & a_{32} & \cdots & a_{3m} \\ 1 & a_{41} & a_{42} & \cdots & a_{4m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \cdot \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} \quad (61)$$

Figure 3: Parallel structure(Causal relation $\Psi_{a_{i\bullet}}$)

which is equivalent to the deterministic Markov operator $\Psi_{a_{i\bullet}} : C_0(\Omega_i) \rightarrow C_0(\Omega_0)$ such that

$$[\Psi_{a_{i\bullet}}(f_i)](\omega_0) = f_i(\psi_{a_{i\bullet}}(\omega_0)) \quad (\forall f_i \in C_0(\Omega_i), \forall \omega_0 \in \Omega_0, \forall i \in 1, 2, \dots, n) \quad (62)$$

Thus, under the identification: $a_{ij} \Leftrightarrow \Psi_{a_{i\bullet}}$, the term "explanatory variable" means a kind of causality. Therefore, we have the observable $O_0^{a_{i\bullet}} \equiv (\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \Psi_{a_{i\bullet}} G_\sigma)$ in $C_0(\Omega_0(\equiv \mathbb{R}^{m+1}))$ such that

$$[\Psi_{a_{i\bullet}}(G_\sigma(\Xi))](\beta) = [(G_\sigma(\Xi))](\psi_{a_{i\bullet}}(\beta)) = \frac{1}{(\sqrt{2\pi\sigma^2})} \int_{\Xi} \exp \left[-\frac{(x - (\beta_0 + \sum_{j=1}^m a_{ij}\beta_j))^2}{2\sigma^2} \right] dx \quad (63)$$

$$(\forall \Xi \in \mathcal{B}_{\mathbb{R}}, \forall \beta = (\beta_0, \beta_1, \dots, \beta_m) \in \Omega_0(\equiv \mathbb{R}^{m+1}))$$

Hence, we have the simultaneous observable $\times_{i=1}^n O_0^{a_{i\bullet}} \equiv (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, \times_{i=1}^n \Psi_{a_{i\bullet}} G_\sigma)$ in $C_0(\Omega_0(\equiv \mathbb{R}^{m+1}))$ such that

$$\begin{aligned} & [(\times_{i=1}^n \Psi_{a_{i\bullet}} G_\sigma)(\times_{i=1}^n \Xi_i)](\beta) = \times_{i=1}^n ([\Psi_{a_{i\bullet}} G_\sigma](\Xi_i))(\beta) \\ &= \frac{1}{(\sqrt{2\pi\sigma^2})^n} \int \cdots \int \exp \left[-\frac{\sum_{i=1}^n (x_i - (\beta_0 + \sum_{j=1}^m a_{ij}\beta_j))^2}{2\sigma^2} \right] dx_1 \cdots dx_n \\ & \quad \times_{i=1}^n \Xi_i \\ & \quad (\forall \times_{i=1}^n \Xi_i \in \mathcal{B}_{\mathbb{R}^n}, \forall \beta = (\beta_0, \beta_1, \dots, \beta_m) \in \Omega_0(\equiv \mathbb{R}^{m+1})) \end{aligned} \quad (64)$$

Assuming that σ is variable, we have the observable $O = (\mathbb{R}^n (= X), \mathcal{B}_{\mathbb{R}^n} (= \mathcal{F}), F)$ in $C_0(\Omega_0 \times \mathbb{R}_+)$ such that

$$[F(\times_{i=1}^n \Xi_i)](\beta, \sigma) = [(\times_{i=1}^n \Psi_{a_{i\bullet}} G_\sigma)(\times_{i=1}^n \Xi_i)](\beta) \quad (\forall \times_{i=1}^n \Xi_i \in \mathcal{B}_{\mathbb{R}^n}, \forall (\beta, \sigma) \in \mathbb{R}^{m+1}(\equiv \Omega_0) \times \mathbb{R}_+) \quad (65)$$

Problem 3 [The generalized linear model] Assume that a measured value $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in X = \mathbb{R}^n$ is obtained by the measurement $M_{C_0(\Omega_0 \times \mathbb{R}_+)}(O \equiv (X, \mathcal{F}, F), S_{[(\beta_0, \beta_1, \dots, \beta_m, \sigma)]})$. We do not know the state $(\beta_0, \beta_1, \dots, \beta_m, \sigma^2)$. Then, from the measured value $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, infer the $\beta_0, \beta_1, \dots, \beta_m, \sigma$! That is, represent the $(\beta_0, \beta_1, \dots, \beta_m, \sigma)$ by $(\hat{\beta}_0(x), \hat{\beta}_1(x), \dots, \beta_m(x), \hat{\sigma}(x))$ (i.e., the functions of x).

Answer. The answer is easy, since it is a slight generalization of Problem 2. Also, it suffices to follow ref. [1]. However, note that the purpose of this paper is to describe Problem 3 (i.e, the quantum linguistic formulation of the generalized linear model) and not to give the answer to Problem 3.

3 Conclusions

Quantum language is clearly defined by the (B), that is,

$$(B_1) \quad \boxed{\text{Quantum language}} = \boxed{\text{Axiom 1}} + \boxed{\text{Axiom 2}} + \boxed{\text{linguistic interpretation}}$$

(=MT(measurement theory)) (measurement) (causality) (how to use Axioms)

Therefore, we do not start from "random variable" but "measurement". Our purpose of this paper was to understand the regression analysis and the generalized linear model in quantum language. In fact, we showed

- (J) the term "explanatory variable in (34) and (59)" is characterized a kind of causality (cf. Figures 2 and 3). And the term "response variable" means the measured value.

We believe that quantum language has a great power of description, and therefore, even statistics can be described by quantum language. We hope that our assertions will be examined from various points of view.

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